Linear Algebra I

28/11/2025, Friday, 18:30 – 20:30

You are **NOT** allowed to use any type of calculators.

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1 Linear equations $2+2+10+3+3=20$ pts

Consider the polynomial $p(x) = a + bx + cx^2$.

(a) Find linear equations in the unknowns a, b, c by using the conditions

$$p(-1) = p(1) = 2 \quad \text{and} \quad p(0) = 1.$$

(b) Write down the augmented matrix.

(c) Put the augmented matrix into the *reduced* row echelon form.

(d) Determine whether the linear equations are consistent or inconsistent. Justify your answer.

(e) Find all solutions.

2 Eigenvalues and diagonalization $4+7+7+7=25$ pts

Let M be the 3×3 matrix given by

$$M = \begin{bmatrix} a & b & c \\ a & b & c \\ a & b & c \end{bmatrix}$$

where a, b, and c are real numbers.

(a) By using the relationship between the determinant and eigenvalues of a matrix, show that 0 is an eigenvalue of M.

(b) By using the definition of eigenvalue, show that $a+b+c$ is an eigenvalue of M.

(c) By using the relationship between the trace and eigenvalues of a matrix, show that the characteristic polynomial of M is $\lambda^2(\lambda - a - b - c)$.

(d) Suppose that $a+b+c \neq 0$. Show that M is diagonalizable. Find a diagonalizer.

3 Subspaces of $\mathbb{R}^n$ $20+5=25$ pts

Let $a, b \in \mathbb{R}$ and

$$S = \left\{ \boldsymbol{x} \in \mathbb{R}^2 \mid (x_1 - ax_2)(x_1 - bx_2) = 0 \right\}.$$

(a) Determine all values of a and b such that S is a subspace of $\mathbb{R}^2$.

(b) Find a basis for S whenever it is a subspace.

4 Cayley-Hamilton theorem $5+15=20$ pts

Let $M \in \mathbb{R}^{3\times 3}$ be such that $\det(M) = 1$ and $\dfrac{-1+\sqrt{3}i}{2}$ is an eigenvalue of M.

(a) Find all eigenvalues of M.

(b) Let $M^{100} = aM^2 + bM + cI_3$. Using Cayley-Hamilton theorem, determine a, b, c.

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10 pts free